第一讲

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黎曼几何 课程主页

staff.ustc.edu.cn/~spliu/Teaching.html

① Michael Spivak I – V

② Jürgen Jost Rie. geom. and geometric analysis 7^{th}

③ do Carmo Rie geometry.

④ Petersen GTM 171

⑤ 伍鸿熙, 沈纯理, 虞言林 黎曼几何初步

⑥ Marcel Berger, A panoramic view of Rie. geometry

助教 李文博

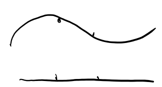
Introduction.

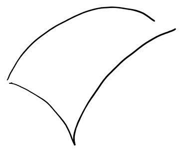
Riemann 1854

On the Hypotheses which lie at the foundation of geometry.

Gauss 1827 General investigations of curved surfaces

Theorema Egregium





Geometry. spaces Gauss equation

Riemann. Spaces. discrete

Differential manifolds (C^∞) + Riemann metric

2-dim $K \equiv 1$ sphere $K \equiv 0$ $\mathbb{R}^2$

$K \equiv -1$

Hilbert 1901: complete

No complete immersed surface in $\mathbb{E}^3$ has constant negative Gauss curvature.

{ All lines in $\mathbb{R}^n$ through the origins }

$n=3$ $\mathbb{R}P^2$

Higher dimensional data.

Manifold

A topological space, M,

(1) every point $x \in M$ has a neighborhood U which is homeomorphic to an open subset $\Omega \subset \mathbb{R}^n$

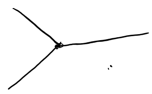
$x: U \longrightarrow \Omega$

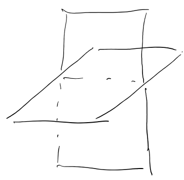
which is homeomorphic to an open subset $\Omega \subset \mathbb{R}^n$

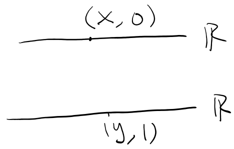
$x: U \longrightarrow \Omega$

Cartan 1927

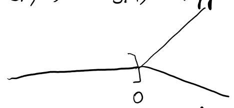
Whitney 1936





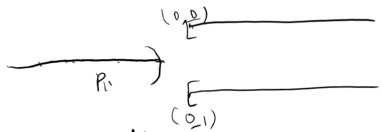


$(x,0) = (y,1)$ iff $x = y \leq 0$



Not locally Euclidean

$(x,0) = (y,1)$ iff $x = y < 0$



locally Euclidean

(2) Hausdorff $\forall\ p, q \quad \exists\ U_p, U_q$ s.t.

$p \in U_p,\ q \in U_q,\ U_p \cap U_q = \emptyset$

(3) M second countable

Thm (Spivak I Appendix A)

For a connected top. space M satisfying (1), (2).

TFAE:

(1) M second countable

(2) M metrizable

(3) M is paracompact

$\forall$ open covering $(\Omega_\alpha)_{\alpha \in A}$ of M

$\exists$ a locally finite covering $(\Omega'_\beta)_{\beta \in B}$ s.t.

$\forall \beta \in B\ \Omega'_\beta \quad \exists\ \alpha \in A$ s.t. $\Omega'_\beta \subset \Omega_\alpha$.

partition of unit

Examples: 1-dim $\mathbb{R}$ (noncpt), S^1 (cpt)

2-dim cpt classification genus orientability

3-dim cpt. Thurston's program

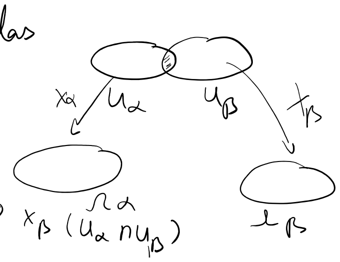
Perelman Ricci flow

Smooth manifold (Differentiable manifold)

$x: U \longrightarrow \Omega$ charts (coordinate neighborhood)

M: $\{U_\alpha, x_\alpha\}$ an atlas

differentiable atlas



$x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \longrightarrow x_\beta(U_\alpha \cap U_\beta)$

C^∞

A maximal differentiable atlas is called a differentiable structure.

Manifold + differentiable structure $\to$ diff. manifold

d-dim. C^∞ mfld

Remark. dim ≤ 3 unique

$\mathbb{R}^4$

Milnor 1956 exotic 7-sphere S^7

Partition of unity.

Lemma. Let M be smooth manifold. $(U_\alpha)_{\alpha \in A}$ is an open covering. Then $\exists$ a partition of unity subordinate to $(U_\alpha)_{\alpha \in A}$.

Denote $(V_\beta)_{\beta \in B}$ be a locally finite refinement

$\varphi_\beta : M \to \mathbb{R}$ $\quad \varphi_\beta \in C_0^\infty(M)$

s.t. (i) $\operatorname{supp} \varphi_\beta \subset V_\beta$, $\forall \beta \in A$

(ii) $0 \leq \varphi_\beta(x) \leq 1$ $\quad \forall x \in M$, $\beta \in B$

(iii) $\sum_{\beta \in B} \varphi_\beta(x) = 1$, $\quad \forall x \in M$.

C^∞ manifold + Riemann metric

vital step: curvature

Metric structure. curve length

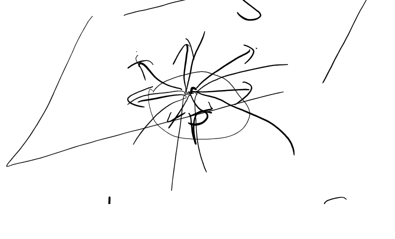
shortest curve between p and q

geodesic

$S \subset T_pM$

2-dim subspace

Consider all $v \in S$



Sectional curvature at p w.r.t S

Plan: (I) Riemannian metric

(II) geodesics

exponential map → normal coordinates

complete: Hopf-Rinow (1931)

(III) Connections, Parallelism, Covariant derivatives

Italian

(IV) Curvature   Second variation

Sectional, Ricci, Scalar

(V) Space forms and Jacobi fields

(VI) Comparison theorems

geometry and topology

(I) Riemannian Metric

§1 Definition: Metric space: Féchet's thesis 1906

M, – C^∞ mfld

$\forall p \in M$, T_pM $\forall V \in T_pM$, need $\|V\|$

T_pM normed space → Finsler geometry

$\|V\|$, $\langle V, W\rangle$ T_pM Hilbert space

inner product

Need to define $\langle V,W\rangle_p$, $\forall$ ~~W~~, W, V $\in T_pM$, $\forall p \in M$

Definition. (Rie. metric) M C^∞ manifold.

A Rie. metric g on M is a "C^∞ assignment":

$\forall$ $p \in M$, T_pM, we assign an inner product

$$g_p(\cdot,\cdot) = \langle \cdot,\cdot\rangle_p$$

which is smoothly dependent on p.

i.e. $f(p) := \langle X_p, Y_p \rangle_p = g_p(X_p, Y_p)$
is a smooth function on $U \subset M$ for any smooth vector fields X, Y on $U \subset M$

<u>local coordinate</u>: $p \in U$ $\{x^1, \dots, x^n\}$

$$x : U \to \Omega$$

T_pM $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$

T_p^*M $\{dx^1, \dots, dx^n\}$

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_p = g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) =: g_{ij}(p)$$

matrix $(g_{ij}(p))_{1 \le i, j \le n}$

$\forall X_p, Y_p \in T_pM$, $X_p = X^i(p) \frac{\partial}{\partial x^i}$, $Y_p = Y^i(p) \frac{\partial}{\partial x^i}$

$$\langle X_p, Y_p \rangle_p = \left\langle X^i(p) \frac{\partial}{\partial x^i}, Y^j(p) \frac{\partial}{\partial x^j} \right\rangle_p$$
$$= \underline{X^i(p)}\ \underline{Y^j(p)}\ g_{ij}(p)$$

tensor $g = g_{ij}\, dx^i \otimes dx^j$

① $g_{ij}(p)$ is smooth on $U \ni p$, $\forall i, j$

② $g_{ij}(p)$, $\forall p \in M$ symmetric, positive definite

M, tensor

<u>Definition'</u>. A Rie metric g on M is a smooth $(0,2)$-tensor satisfying

$$g(X,Y) = g(Y,X), \quad g(X,X) \ge 0,$$

$$g(X,X) = 0 \iff X \equiv 0$$

for any C^∞ vector fields X, Y

<u>Riemannian manifold</u>: (M, g)

$M = \mathbb{R}^n$ $\quad T_p\mathbb{R}^n \cong \mathbb{R}^n$

Riem metric: $g(X,Y) = \underline{X^T Y}$ $\quad X^T A Y$

matrix $(g_{ij}) = (\delta_{ij})$ $\quad A$

Induced metric: Let $f: M^n \longrightarrow N^{n+k}$ be a C^∞ immersion (i.e. $df_p: T_pM^n \to T_{f(p)}N$ injective $\forall p \in M$)

If (N, g_N) be a Rie. mfld.

Define the pull-back metric $f^* g_N$ on M:

$$(f^* g_N)_p (X_p, Y_p) := \underline{(g_N)_{f(p)} (df_p(X_p), df_p(Y_p))}$$

$\forall X_p, Y_p \in T_pM$.

. Verify it is indeed a Rie. metric

$$\underline{(f^* g_N)_p (X_p, X_p) = 0} \; \overset{?}{\Rightarrow} \; X_p = 0$$

$$df_p(X_p) = 0$$

$M \subset N$ is a immersed submanifold.

inclusion $i: M \longrightarrow (N, g_N)$ immersion

$i^* g_N$:

$g_N : T_pN \qquad T_pM \subset T_pN$

Product metric :

(M, g_M) , (N, g_N) , $M \times N$

projection: $\pi_1 : M \times N \longrightarrow M \qquad \pi_1((p,q)) = p$

$\pi_2 : M \times N \to N$

$\forall \; (p, q) \in M \times N, \quad \forall X, Y \in T_{(p,q)}(M \times N)$

$$g_{(p,q)}(X, Y) := (g_M)_p (d\pi_1(X), d\pi_1(Y)) + (g_N)_q (d\pi_2(X), d\pi_2(Y))$$

(S^1, g_{S^1})

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n} \qquad (T^n, g_{T^n})$$

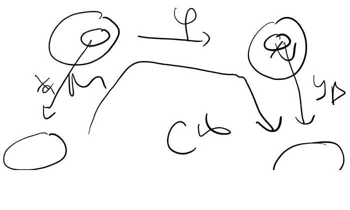
Remark: (M, g)

Definition. (Isometry) Let (M, g_M), (N, g_N).

Let $\varphi: M \to N$ be a diffeomorphism

If $\varphi^* g_N = g_M$

then we call φ is



an isometry:

$\forall\ X, Y \in T_p M, \ \forall\ p \in M$

$$\varphi^* g_N(X,Y) = g_M(X,Y)$$

Existence of Riemannian Metric

Thm A C^∞ mfld M has a Riem. metric.

Proof: ·) Whitney 1936: C^∞ mfld $M^n \longrightarrow \mathbb{R}^{2n+1}$

·) Intrinsic: partition of unity.

$\{U_\alpha\}_{\alpha \in A}$ locally finite covering

$\{\varphi_\alpha\}_{\alpha \in A}$ subordinate to $\{U_\alpha\}_{\alpha \in A}$

On each U_α, $x_\alpha : U_\alpha \to \Omega \subset \mathbb{R}^n$
diffeomorphism $\{x_\alpha^1, \dots, x_\alpha^n\}$

$$g_\alpha = \sum_{i} a_{ij}\, dx_\alpha^i \otimes dx_\alpha^i$$

~~$\forall p \in$~~ $$g = \sum_\alpha \varphi_\alpha g_\alpha$$

$\forall p \in M$, $\forall X, Y \in T_p M$, $g_p(X,Y) := \sum_\alpha \varphi_\alpha(p) (g_\alpha)_p (X,Y)$

Verify g is indeed a Rie. metric.

Exercise. $\square$

下课.